

# MAGNETOROTATIONAL TURBULENCE TRANSPORTS ANGULAR MOMENTUM IN STRATIFIED DISKS WITH LOW MAGNETIC PRANDTL NUMBER BUT MAGNETIC REYNOLDS NUMBER ABOVE A CRITICAL VALUE

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## ABSTRACT

The magnetorotational instability (MRI) may dominate outward transport of angular momentum in accretion disks, allowing material to fall onto the central object. Previous work has established that the MRI can drive a mean-field dynamo, possibly leading to a self-sustaining accretion system. Recently, however, simulations of the scaling of the angular momentum transport parameter  $\alpha_{SS}$  with the magnetic Prandtl number  $Pm$  have cast doubt on the ability of the MRI to transport astrophysically relevant amounts of angular momentum in real disk systems. Here, we use simulations including explicit physical viscosity and resistivity to show that when vertical stratification is included, mean field dynamo action operates, driving the system to a configuration in which the magnetic field is not fully helical. This relaxes the constraints on the generated field provided by magnetic helicity conservation, allowing the generation of a mean field on timescales independent of the resistivity. Our models demonstrate the existence of a critical magnetic Reynolds number  $Rm_{crit}$ , below which transport becomes strongly  $Pm$ -dependent and chaotic, but above which the transport is steady and  $Pm$ -independent. Prior simulations showing  $Pm$ -dependence had  $Rm < Rm_{crit}$ . We conjecture that this steady regime is possible because the mean field dynamo is not helicity-limited and thus does not depend on the details of the helicity ejection process. Scaling to realistic astrophysical parameters suggests that disks around both protostars and stellar mass black holes have  $Rm \gg Rm_{crit}$ . Thus, we suggest that the strong  $Pm$  dependence seen in recent simulations does not occur in real systems.

*Subject headings:* MHD: Instabilities

## 1. INTRODUCTION

Accretion disks form in a wide variety of astrophysical objects, from active galactic nuclei to dwarf novae and protostars. In the former two, disks are responsible for the tremendous luminosities, while in the latter they are the site of planet formation. Understanding the structure and evolution of disks is therefore critical to understanding each of these objects. The successful Shakura & Sunyaev (1973) model of accretion disks relies on viscous transport of angular momentum outward through the disk, allowing mass to spiral inwards. The amount of viscosity necessary to explain observations requires turbulence, as molecular viscosity is far too small. Shakura & Sunyaev (1973) parametrized the turbulent stress providing this viscosity by  $\alpha_{SS} = (\langle \rho u_x u_y \rangle - \langle B_x B_y \rangle) / P_0$  where  $P_0$  is the midplane pressure of the disk. The magnetorotational instability (MRI) offers the most viable mechanism for providing these enhanced levels of angular momentum transport (Balbus & Hawley 1998). It is known to saturate in a magnetohydrodynamic turbulent state that transports angular momentum outward through the disk.

Aside from its central role in accretion disk theory, the MRI also provides an interesting system in which to study mean field dynamo theory. Starting from a zero-net flux magnetic field configuration, the MRI acts as a mean field dynamo, generating strong, fluctuating fields with order at wavelengths as large as the simulation box (Brandenburg et al. 1995; Hawley et al. 1996). The system, fed by the free energy from Keplerian shear, can sustain angular momentum transport with only a weak (sub-equipartition) field and the presence of a negative radial gradient in angular velocity. Because the turbulence is driven by the MRI itself, the Lorentz force is essential to the operation of the dynamo, ensuring that the system is never in a kinematic phase (Hawley et al. 1996).

In recent years, the MRI has become the target of intense numerical investigation owing to recent studies showing a decline of the angular momentum transport rate with increasing resolution (Fromang & Papaloizou 2007; Pessah et al. 2007) in the simplest 3D systems that demonstrate MRI turbulence: unstratified, periodic, shearing boxes. Lesur & Longaretti (2007) and Fromang et al. (2007) showed that this decline can be described as a rather steep power-law dependence of transport on the magnetic Prandtl number,  $Pm \equiv \nu/\eta$ , the ratio of viscous momentum diffusion to resistive magnetic diffusion. Given that real astrophysical disks, especially

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protoplanetary disks, have extremely low  $\text{Pm} \sim 10^{-8}$  (Balbus & Henri 2008), this would imply that the MRI could not be responsible for angular momentum transport in such systems.

Subsequently, several authors have attempted to explain these angular momentum scaling results, and in doing so suggest why they may not be applicable to accretion disks. These include the idea that unstratified shearing boxes lack a characteristic outer scale (Vishniac 2009); that the initial magnetic field strength in the Fromang et al. (2007) simulations is too weak and they are thus stable to non-axisymmetric MRI modes that are essential to sustained transport (Kitchatinov & Rüdiger 2010); or that an unstratified shearing box with periodic boundary conditions cannot sustain large scale dynamo action, and small-scale dynamo action is known to be Pm-dependent, thus rendering the magnetic field necessary for the MRI susceptible to similar Pm-dependence (Käpylä & Korpi 2010). A pair of recent papers has demonstrated that even without explicit viscosity and resistivity, *stratified* MRI simulations *do* converge to a consistent value of  $\alpha_{SS}$  with increasing resolution (Davis et al. 2010; Shi et al. 2010). This strongly suggests that stratification plays a major role in the dynamics and saturation of the MRI, and demands a thorough investigation of how it does so, especially in the case of a zero-net flux field, where dynamo action is inextricably linked to continued accretion.

In the absence of an externally imposed mean field, the shearing box system can completely destroy the initial magnetic field. For sustained turbulence to be possible, the MRI must create field by dynamo action. This field in turn allows the continued excitation of new MRI modes. Thus, the zero-net flux shearing box is a non-linear system balancing fluctuating, dynamo-generated fields with MRI generated turbulence. In order to understand the angular momentum transport from MRI turbulence and how it scales with the physical parameters of the system, we need to understand the underlying dynamo operates.

In this paper, we focus our analysis on the question of dynamo action in the MRI and attempt to understand the connection between mean field dynamos and the amount of angular momentum transport. We include explicit viscosity and resistivity in order to consider the role of the viscous and magnetic Reynolds numbers  $\text{Re}$  and  $\text{Rm}$  in stratified disks. By considering a broader range of  $\text{Pm}$  than previous studies, we find evidence for the existence of a critical magnetic Reynolds number in *stratified* disks. This  $\text{Rm}_{\text{crit}}$  may represent a boundary for sustained dynamo activity, which in turn controls angular momentum transport. We note two dynamo behaviors, one corresponding to a sustained, organized dynamo, and the other corresponding to a transient, chaotic dynamo. These behaviors manifest above and near the critical  $\text{Rm}_{\text{crit}}$ , respectively. We then attempt to connect the properties of the stratified MRI to other, better studied dynamo systems at high and low  $\text{Pm}$ . Specifically, we focus on the conservation of magnetic helicity, and attempt to understand why the MRI is able to build mean fields on a relatively rapid timescale compared with other mean field dynamos. We find that the MRI generated dynamo fields are not significantly helical at any time during their evolution, and this fact explains why their growth is not limited by the resistive timescale as

might be expected for fully helical fields.

This study is most similar to the work of Käpylä & Korpi (2010), in that we consider the effects of boundary conditions on the transport of magnetic helicity and its relation to the generation of large-scale magnetic fields. However, we consider the effect of varying  $\text{Pm}$  on stratified shearing boxes. Furthermore, we attempt to understand why the *stratified* but *periodic* simulations of Davis et al. (2010); Shi et al. (2010) show convergence with resolution. Because of their periodic boundary conditions, they cannot support the same type of flux-transport dynamo action that renders angular momentum transport independent of  $\text{Pm}$  in the simulations of Käpylä & Korpi (2010). It is our conclusion that, despite the presence of a large scale dynamo, the large scale field is not helicity limited, though the MRI does eject helicity through open boundaries if they are present. This explains why the MRI dynamo can operate in periodic boxes, and points toward a theoretical understanding of the non-linear shearing box system. Additionally, the dynamo period does not show a discernible dependence on  $\text{Pm}$ . Most importantly, we find that the angular momentum transport coefficient appears independent of  $\text{Pm}$  *as long as the magnetic Reynolds number  $\text{Rm}$  remains above a critical value,  $\text{Rm}_{\text{crit}} \sim 3000$ .*

In § 2, we review our methods. We present the scaling results in § 3, followed by a discussion of the importance of magnetic helicity conservation for these results in § 4. We conclude and note several avenues for future work in § 5.

## 2. METHODS

Using the shearing box formalism (Hawley et al. 1995; Regev & Umurhan 2008), we study a stratified patch of a Keplerian accretion disk threaded by an initial magnetic field of the form  $\mathbf{B}_0 = B_0 \sin(x) \mathbf{e}_z$  with maximum midplane value of plasma  $\beta = 2c_s^2/v_A^2 \simeq 89$ . We solve the equations of isothermal, compressible magnetohydrodynamics using the Pencil Code<sup>2</sup> (Brandenburg & Dobler 2002; Johansen et al. 2009), a spatially sixth-order, temporally third-order finite difference method. The constraint  $\nabla \cdot \mathbf{B} = 0$  is enforced by solving the evolution equation for the magnetic vector potential,

$$\frac{\partial \mathbf{A}}{\partial t} + u_y^0 \frac{\partial \mathbf{A}}{\partial y} = \mathbf{u} \times \mathbf{B} + \frac{3}{2} \Omega A_y \hat{\mathbf{e}}_x + \eta \nabla^2 \mathbf{A} + \eta_3 \nabla^6 \mathbf{A} + \nabla \phi_{\text{kep}}, \quad (1)$$

where  $\eta_3 \nabla^6 \mathbf{A}$  is a hyperdiffusion operator to dissipate excess energy at the grid scale, and the second terms on each side are from the shearing box formalism. We use similar hyperdiffusion terms on all dynamical equations (see Johansen & Klahr 2005; Oishi et al. 2007). The value of  $\eta_3$  given in Table 1 is chosen so that the hyperdiffusive Reynolds numbers are roughly unity at the Nyquist scale, where  $\text{Re}_{Ny} = u_{\text{rms}}/\eta_3 k_{Ny}^5$ . the magnitude of the hyperdiffusion operators is chosen for numerical stability and speed, and has no significant effects on our results. The hyperdiffusivity scales inversely with the grid size  $dx$ , so the convergence seen in our high resolution runs suggests that  $\text{Rm}_{\text{crit}}$  is unlikely to be affected by the value of the coefficient.

<sup>2</sup> <http://www.nordita.org/software/pencil-code/>

In order to study the effects of Pm on the saturated MRI turbulence, we vary the viscosity  $\nu$  and resistivity  $\eta$ . We define  $\text{Re} \equiv SH^2/\nu$  and  $\text{Rm} \equiv SH^2/\eta$ , where  $S = \Omega d \log \Omega / d \log r = q\Omega = -3/2\Omega$  is the (Keplerian) shear rate in the box in units of rotation  $\Omega$  and  $H^2 = c_s^2/\Omega^2$  is the scale height of the disk. Note that our definition of  $S$  is opposite in sign with respect to Lesur & Longaretti (2007); there is no consistent definition. We set  $c_s = H = \Omega = \mu_0 = 1$ , choosing our units to minimize the number of values we need to remember. In these units,  $\text{Re} = 1.5/\nu$ ,  $\text{Rm} = 1.5/\eta$ , and  $B_0 = 0.15$ . The advantage of defining Re and Rm this way is that they can be set *a priori*, though they do not measure the relative effects of advection and dissipation in the saturated state of the MRI. Because our Re and Rm are *a priori* parameters, we have checked that they correlate with the ratio of turbulent advection to dissipation,  $\text{Re}' = u_{rms}/k_1\nu$  and  $\text{Rm}' = u_{rms}/k_1\eta$ , respectively. Here,  $k_1 = 2\pi/L_z$  is the smallest integer wavenumber in the box. Note that  $k = k_1/2$  is consistent with the vertical field boundary conditions, and large-scale fields of this size can also fit within the box. This data is presented in Table 1. Indeed, in Figure 1 Rm is particularly well correlated with  $\text{Rm}'$ .

Table 1 summarizes the parameters of our models. We report resolution in terms of zones per scale height, with a standard resolution of 64 zones/H, though we also ran three simulations with 128 zones/H to confirm the convergence of our results. All of our runs use cubic zones,  $dx = dy = dz$  and were run for  $100t_{orb}$ . Our standard box size is  $1H \times 4H \times 4H$ .

Our simulations are periodic in  $y$  (azimuthal), shearing periodic in  $x$  (radial), and one of three different choices for  $z$  (axial): periodic, perfect conductor, or vertical field (hereafter VF). Among these choices, the first does not allow a flux of magnetic helicity out of the simulation domain, while the others do. This has a significant effect on the resulting dynamo action and turbulence, though not nearly as dramatic as in the unstratified results of Käpylä & Korpi (2010). Vertical field boundary conditions ensure  $B_x = B_y = 0$  on the upper and lower boundaries. They have been used before in a number of accretion disk (e.g. Brandenburg et al. 1995; Ziegler & Rüdiger 2001; Käpylä & Korpi 2010; Gressel 2010) and magnetoconvection (Hurlburt & Toomre 1988) simulations as a simplified version of the vacuum boundary conditions appropriate to the surface of a disk or star. The total magnetic flux in this case is not conserved, but is free to grow or decay in the  $B_x$  and  $B_y$  components, thus allowing a mean field dynamo action within the domain in a computationally convenient manner, though they provide a somewhat artificial constraint on the field at the boundaries. They are known to produce spurious current density near the boundaries (Ziegler & Rüdiger 2001), but this effect does not significantly affect bulk properties of the flow, as we show below. The vertical field conditions are implemented in our vector potential code by setting  $\partial_z A_x = \partial_z A_y = A_z = 0$  at the  $z$  boundaries.

### 3. RESULTS

Here, we present the main results from the suite of simulations described in Table 1.

#### 3.1. Scaling with Pm and Rm

Figure 2 shows that the dependence of  $\alpha_{SS}$  on Pm is not well described by a single power-law at a given Re, as had been previously claimed for unstratified shearing boxes by Lesur & Longaretti (2007) and Fromang et al. (2007). The points are the mean  $\alpha_{SS}$  for all times  $t > 20t_{orb}$ , and the error bars represent the standard deviation over that range. These runs all use the VF boundary conditions. The figure shows evidence of a cut-off that moves to higher Pm as Re decreases. This is indicative of a critical Rm, most visible for  $\text{Re} = 3200$  (center left) and  $\text{Re} = 9600$  (lower left). Above  $\text{Rm}_{crit}$ , it appears that  $\alpha_{SS}$  is consistent with a constant value. These data show that the behavior of the stratified, zero net flux MRI system has two distinct regimes, controlled by  $\text{Rm}_{crit}$ . When  $\text{Rm} < \text{Rm}_{crit}$ , the transport is strongly dependent on Pm; when  $\text{Rm} > \text{Rm}_{crit}$ , the transport is independent of Pm. Given that astrophysical disks have typical  $\text{Re} \gtrsim 10^{16}$ , this result means that the MRI is capable of robust angular momentum transport even at low Pm, so long as  $\text{Rm} > \text{Rm}_{crit}$ . We discuss these estimates in more detail in § 5 and suggest that this criterion is easily met in most disks.

We recast these results in the (Rm, Re) plane in Figure 3. This figure makes clear the fact that along with a reduced but still-present Pm-dependence, there is a fairly clear critical Rm: transport is significantly reduced left of the vertical line near  $\text{Rm} \sim 3000$ . As the simulations approach  $\text{Rm}_{crit}$ , the  $\alpha_{SS}$  variance increases significantly.

#### 3.2. Two Dynamo Behaviors

Figure 4 shows that the MRI appears to operate in two distinct dynamo states, one in which regular  $\langle B_y \rangle$  cycles appear with a characteristic period of  $\tau_B \sim 10t_{orb}$ , and one with irregular variations with timescales on the order of  $\sim 50t_{orb}$ . The key control parameter appears to be Rm, as the second, third, and fourth panels from the top show the regular behavior despite being at different Pm. There is no discernible trend in cycle period with Pm, though the top panel, with  $\text{Rm} = 12800, \text{Pm} = 2$  seems to show an intermediate behavior. However, all runs with  $\text{Rm} \lesssim 3200$  conclusively show the irregular behavior. There appears to be a secondary Pm effect as well, since the two  $\text{Rm} = 3200$  models show different behavior depending on Pm (third and fifth panels from the top).

These two drastically different behaviors complicate efforts to ascertain scaling properties of the MRI as a function of dimensionless parameters. The range of available Pm is limited by numerical resolution both from above and below: large values of Pm imply a viscous scale much larger than the resistive one, while small Pm implies the opposite. Since both length scales must fit on (and be resolved by) the grid, and indeed neither can be so large as to stabilize the largest linear MRI modes that fit in the simulation box, our parameter range is necessarily limited. Furthermore, since low Rm runs transition to a very different behavior with much larger cycle period and very different turbulent properties in a discontinuous fashion, our range of available Pm space for the regular dynamo mode is further limited.

Thus, in what follows, we briefly describe the irregular dynamo before focusing on connecting the details of

the MRI turbulence in the regular region of parameter space to better-established results on small and large-scale dynamo action. The MRI is known to produce both small and large scale dynamo action, the latter typically requiring stratification or open boundary conditions (though Lesur & Ogilvie (2008) demonstrate a large scale dynamo with neither). The small-scale dynamo for driven, isotropic, homogeneous, incompressible turbulence has  $\text{Rm}_{\text{crit}}$  that *increases* with *decreasing*  $\text{Pm}$  (Schekochihin et al. 2005), and thus it behooves us to understand how MRI turbulence fits into this picture.

### 3.3. Irregular Regime

When  $\text{Rm}$  drops below  $\text{Rm}_{\text{crit}}$ , the mean field dynamo switches to a irregular regime, showing quasi-periodic magnetic cycles, with longer cycle lengths (one hesitates to call them periods)  $\sim 40t_{\text{orb}}$  (see the lower two panels of Figure 4). This is consistent with Figure 15 of Davis et al. (2010), who also showed long cycle lengths in highly resistive simulations. In this regime, the turbulence often appears only in one half-plane, either  $z > 0$  or  $z < 0$ , sometimes staying that way for nearly the duration of the simulation. This kind of behavior at first appears unphysical, as the stratified shearing box system, while having odd parity about the midplane, should not necessarily damp perturbations of one helicity more than the other. Indeed, in all simulations, the kinetic helicity shows erratic fluctuations of sign with respect to the midplane.

However, this is explainable in a simple  $\alpha - \Omega$  treatment. The only prerequisite for this explanation is that the two half-planes be dynamically decoupled from one another. For the case of VF boundary conditions,  $B_y(z = \pm L_z) \mapsto 0$ . The largest wavenumber modes compatible with this boundary condition are  $B_y \propto \sin(2\pi/L_z z)$  and  $B_y \propto \sin(\pi/L_z z)$ , corresponding to  $k = 1$  and  $1/2$ , respectively. The  $\alpha - \Omega$  dynamo operates when the mean induction equation takes the form

$$\partial_t \langle \mathbf{B} \rangle = \alpha \langle \mathbf{B} \rangle - q\Omega \langle B_x \rangle \hat{\mathbf{y}} + \eta \nabla^2 \langle \mathbf{B} \rangle. \quad (2)$$

If during a cycle period, the MRI turbulence shuts off, the  $\alpha$  effect that it produces will cease, leaving a mean induction equation that looks like

$$\partial_t \langle B_x \rangle = \eta \nabla^2 \langle B_x \rangle \quad (3)$$

for the  $x$  component and

$$\partial_t \langle B_y \rangle = -q\Omega \langle B_x \rangle + \eta \nabla^2 \langle B_y \rangle \quad (4)$$

for the  $y$  component. Because the decay time for modes is roughly  $t_{\text{decay}} \simeq 1/k^2\eta$ , small scale structure will undergo selective decay, leaving only the largest scale modes, the decay time of which is  $\sim 108t_{\text{orb}}$  for  $\text{Rm} = 1600$ . At this stage,  $\langle B_y \rangle$  has about this much time to grow linearly via stretching of any residual  $\langle B_x \rangle$  to amplitudes at which non-axisymmetric MRI can reestablish fluid turbulence and hence an  $\alpha$  effect. If the two half-planes are not strongly coupled, it is possible that the turbulence will die out in one half of the box first, leading to an  $\alpha$  effect only in the upper or lower midplane. This scenario is essentially the same as the one demonstrated in the midplane by Simon et al. (2011).

### 3.4. Dynamo coefficients

Understanding the origin of the scaling of angular momentum transport with magnetic Prandtl number requires understanding the underlying field generation mechanism. Gressel (2010) has demonstrated that the field patterns present in the MRI can be explained in terms of a mean-field dynamo model that includes dynamical  $\alpha$ -quenching, a mechanism that modulates  $\alpha$  by enforcing magnetic helicity conservation as the mean field grows (see § 4.2 for more details). Here, we decompose fields from the simulations into their mean and fluctuating components,  $\mathbf{B} = \bar{\mathbf{B}} + \mathbf{b}$ , where  $\bar{\mathbf{B}}$  is a horizontal  $(x - y)$  average and  $\mathbf{b}$  is the fluctuating field. We denote full box averages as  $\langle \mathbf{B} \rangle$ . In the standard  $\alpha - \Omega$  dynamo mechanism, the  $\alpha$  effect of isotropic, helical turbulence generates poloidal field from toroidal fields, which are in turn sheared out by differential rotation  $\Omega$  and regenerate toroidal field, thus leading to exponential amplification. Based on the early work of Brandenburg et al. (1995), the traditional  $\alpha - \Omega$  scenario can explain the observed periodic dynamo generation and propagation of  $\bar{B}_y$  away from the disk midplane if the  $\alpha$  term has the opposite sign of that expected from rotating, stratified turbulence (the strong Keplerian shear has no problem stretching poloidal field to generate toroidal field; the issue at hand is generating the former). However, that expected sign was derived from an analysis that only assumed the presence of a kinetic  $\alpha_K$  effect from the helicity of the fluid turbulence. Once the magnetic  $\alpha_M$  from the Lorentz force is also included (Pouquet et al. 1976) (see also § 4.2), the total  $\alpha = \alpha_K + \alpha_M$  is dominated by  $\alpha_M = 1/3\tau \langle \nabla \times \mathbf{v}_a \cdot \mathbf{v}_a \rangle$ , where  $\tau$  is a typical turbulent correlation time and  $\mathbf{v}_a = \mathbf{b}/\sqrt{4\pi\rho}$ . The total  $\alpha$  has the required sign (Gressel 2010). While we similarly find that  $\alpha_M \simeq 10\alpha_K$ , our results for the  $z$  profile of  $\alpha_M$  (and thus the total  $\alpha$ ) appear to contradict those of Gressel (2010). Figure 5 shows the  $\alpha_M$  profile for three simulations with  $\text{Pm} = 1, 4$  and  $\text{Re} = 3200, 6400, 12800$ . There is no monotonic trend with  $\text{Pm}$ , and there are some differences in shape, but the overall profile is negative in the upper plane ( $z > 0$ ) and positive in the lower, opposite to that found by Gressel (2010). However, it is worth noting that both the  $\text{Re} = 12800$  and the  $\text{Re} = 3200$  runs show a reversal of the  $\alpha_M(z)$  profile near the midplane, just as found by Gressel (2010), though with the opposite sign.

The only significant differences between his study and ours are the vertical boundary conditions on the fluid, which are outflow in his case, and the fact that his domain covers  $6H$ , while our domains typically cover  $4H$ . We have run one simulation, with  $\text{Re} = 6400$  and  $\text{Pm} = 4$ , with a vertical domain of  $6H$ . Figure 6 shows  $\alpha_M(z)$  for both the standard and extended domain for this model. The figure suggests a possible explanation for the discrepancy between our models and Gressel's. When we enlarge our domain, we see regions with  $|z| \gtrsim 2$  showing a positive  $\alpha_M$  effect, while the regions  $|z| \lesssim 2$  are roughly the same between the  $4H$  and  $6H$  models, outside of narrow boundary layers in both cases. Thus, our models just appear to show a stronger reversal of the sign of  $\alpha_M$  near the midplane than does Gressel (2010). Given that buoyancy and Parker instabilities become more prominent with height, it seems that the most

likely explanation for the discrepancy between our results and Gressel's is a combination of a reduced domain for our results and the outflow fluid boundary conditions in Gressel's work. The outflow boundary conditions allow buoyant fluid to escape out of the top of the box, while our closed, stress free lids force that flow to recirculate. Thus, we do not consider our apparently discrepant results to actually be evidence against the standard  $\alpha - \Omega$  mechanism acting to drive the observed dynamo phenomenology.

#### 4. DISCUSSION

##### 4.1. Comparison with previous work

In order to place our results in the context of other recent studies, we begin by comparing our results to the unstratified results presented by Käpylä & Korpi (2010). These authors performed two series of unstratified shearing-box simulations of the MRI, both initialized with zero-net magnetic flux in the  $z$ -direction. One set of simulations had periodic boundary conditions on the magnetic field on the  $z$  boundary, the other had VF boundary conditions. Käpylä & Korpi (2010) show that in the latter case, the angular momentum transport coefficient  $\alpha_{SS}$  is *independent* of  $\text{Pm}$ , while with periodic boundary conditions,  $\alpha_{SS} \propto \text{Pm}^2$ . We find that even with periodic boundary conditions, a stratified disk with high enough  $\text{Rm}$  has  $\alpha_{SS}$  basically independent of  $\text{Pm}$ .

We are not the first to suggest a critical  $\text{Rm}$  for MRI turbulence. In the unstratified case, Fleming et al. (2000) and Sano & Stone (2002) both noted the existence of a sharp cutoff in  $\alpha_{SS}$  when the resistivity exceeded a certain value. The latter study used the Elsasser number,  $\Lambda = v_{A0}^2/\eta\Omega$  (though they refer to this as  $\text{Rm}$ ). However, aside from using  $\Lambda$ , their results also measure the  $\alpha_{SS}$  cutoff as a function of the *initial* magnetic field strength, for both net flux and zero-net flux initial fields. In our case, we do not make reference to the initial field strength; instead, by using the instantaneous value of  $\text{Rm}$  as our parameter, we measure the relative action of shear against dissipation. This is the appropriate measure for a non-linear, self-sustained system such as the stratified, zero-net flux MRI that we study here, because any dynamo-generated energy must ultimately come from the free energy in the shear flow.

More recently, a controversy regarding the role of channel modes in saturating the unstratified MRI when a net  $z$  field is present has led to similar results regarding the  $\text{Rm}_{\text{crit}}$  for MRI-driven transport. Pessah (2010) demonstrated the existence of a critical  $\Lambda$  below which the angular momentum transport by the MRI dropped off dramatically. Numerical simulations of the same problem by Longaretti & Lesur (2010), though seemingly disproving the Pessah (2010) saturation mechanism, also suggests a trend similar to the one we report, with two regimes separated by  $\text{Pm} \sim 1$ . With  $\text{Pm} < 1$ , the transport scaling is dominated by  $\text{Rm}$ , while above it the scaling is mostly with  $\text{Pm}$ . Their primary aim was to elucidate the role of axisymmetric channel modes in the saturation of MRI turbulence in the presence of a net  $z$  field. In our study, there are no channel modes, as there is no net  $z$  field, and it is not too surprising that our results are slightly different. Indeed, the physics our control parameter is attempting to represent is not the growth rate and wave

number of a (quasi-)linear channel mode, but instead a turbulent dynamo that tends to create a net *toroidal* field ( $y$ -field in our simulations), on which any MRI growth is non-axisymmetric and thus transient though with extremely high growth factors (Balbus & Hawley 1992).

##### 4.2. Dynamo Action and Current Helicity

We interpret our results in light of the ability of the system to build up a large scale magnetic flux. Our results suggest that, with stratification, the MRI can act as a mean-field dynamo with any boundary conditions, unlike the unstratified case. In order to clarify terms, we will refer to a mean-field dynamo as any system capable of building magnetic fields at the lowest possible wavenumber in the box, either  $k = 1$  in the periodic case or  $k = 0$  (a true mean field) in the case including a VF boundary condition.

In the modern picture of mean-field dynamo theory, ensuring the conservation of magnetic helicity,

$$H \equiv \langle \mathbf{A} \cdot \mathbf{B} \rangle, \quad (5)$$

provides an important constraint on the evolution of the mean field. Magnetic helicity conservation provides both a compelling physical mechanism for the existence of mean field dynamos as well as quantitative predictions for saturated field strengths and the timescales necessary to reach those strengths. We briefly sketch out some background related to magnetic helicity conservation and then apply it to our MRI simulations.

The evolution equation for  $H$  can be written

$$\frac{dH}{dt} = -2\eta \langle \mathbf{J} \cdot \mathbf{B} \rangle - 2 \oint (\mathbf{A} \times \mathbf{E} + \phi \mathbf{B}) \cdot \hat{\mathbf{n}} dS, \quad (6)$$

where Ohm's law gives  $\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J}$  and  $\phi$  is an arbitrary gauge term in the definition of the vector potential (Brandenburg & Subramanian 2005). In our simulations we use the Kepler Gauge,  $\phi = \mathbf{U}^{\text{kep}} \cdot \mathbf{A}$ ; see § A for more details as well as Brandenburg et al. (1995). The first term is due to resistivity acting on the current helicity  $C \equiv \langle \mathbf{J} \cdot \mathbf{B} \rangle$ , while the second is a boundary flux term that can entirely determine the behavior of the MRI in unstratified simulations (Käpylä & Korpi 2010). In the stratified case, the vertical boundary conditions are considerably less important. This is evident in the periodic, stratified simulations of Davis et al. (2010), who find sustained turbulence for lower  $\text{Pm}$  than similar unstratified boxes. We note that our gauge choice eliminates all possible horizontal magnetic helicity fluxes (see Hubbard & Brandenburg 2011, for a detailed explanation), leaving only the possibility of vertical fluxes, which could be driven by a turbulent diffusivity or shear, as in the case of Vishniac & Cho (2001).

By using an eddy-damped, quasi-normal, Markovian closure scheme, Pouquet et al. (1976) demonstrated that current helicity drives a magnetic  $\alpha_M$  effect, akin to the fluid  $\alpha$  effect of classical mean field dynamo theory. This  $\alpha_M$  in turn leads to an inverse cascade of magnetic energy, and thus the build up of large scale magnetic field. That inverse cascade can be seen simply as a result of the conservation of magnetic helicity. We outline this process following Brandenburg & Subramanian (2005). Consider a fully helical field, which has its scale-

dependent relative helicity,

$$\mathcal{H}(k) = \frac{kH(k)}{2M(k)} = 1. \quad (7)$$

$H(k)$  and  $M(k)$  are the Fourier transforms of magnetic helicity and magnetic energy, respectively.  $\mathcal{H} = 1$  implies  $kH(k) = 2M(k)$ . The non-linear terms allow coupling only among modes  $k$  and  $p$  satisfying  $p + q = k$ , where  $q$  is an arbitrary mediating wavenumber in the three-wave interaction, helicity is conserved for each mode, so  $pH(p) = 2M(p)$  and  $qH(q) = 2M(q)$ . Since energy is conserved in the interaction,  $M(k) = M(p) + M(q)$ , and thus  $pH(p) + qH(q) = kH(k)$ . Because helicity must also be conserved,  $H(k) = H(p) + H(q)$ , and

$$k = \frac{pH(p) + qH(q)}{H(p) + H(q)}. \quad (8)$$

If  $k$  is the final wavenumber, one of  $p$  or  $q$  is the starting wavenumber, and Equation 8 demands that  $k$  is less than or equal to the maximum of  $p$  or  $q$ . Since  $k$  is smaller than or equal to the parent wavenumber, this corresponds to an inverse cascade of magnetic energy. Pouquet et al. (1976) explicitly relate this to the current helicity, and construct a magnetic  $\alpha_M$  term that back-reacts on the flow as this cascade proceeds.

This formalism gives a heuristic explanation for the existence of a mean field dynamo driven by small-scale, helical flows, and incorporating the magnetic helicity constraint into mean field dynamo models allows them to correctly predict the saturation behavior of  $\alpha^2$  dynamos (Brandenburg 2001). Here, we want to understand the MRI dynamo action in our simulations in light of the constraints imposed by helicity conservation. We explore both boundary terms and scale transfer of  $H$ .

We begin by testing the Pencil Code's numerical conservation of magnetic helicity. For a system with periodic boundary conditions, the second term in Equation 6 is zero, and the only contribution to changes in helicity can come from resistively limited current helicity fluctuations. Therefore, if our simulations numerically conserve helicity, a run with periodic boundary conditions should show helicity variations only on a resistive timescale. In order to establish that helicity conservation is robust in our simulations, we ran a simulation with  $\text{Re} = 3200$  and  $\text{Pm} = 2$ , with periodic boundary conditions, and tracked the time evolution of magnetic and current helicities (upper panel of Figure 7). We used a simple forward Euler scheme to integrate  $dH/dt = -2\eta C$  with volume average data from the simulations. The  $H(t)$  that results from this integration is shown in Figure 7 as the green triangles, while the  $H$  calculated directly during the run is given by the blue solid line. The agreement is quite good, considering the crudeness of the integration method and the sparseness of the data (it is only sampled at intervals of 100 timesteps). As a result, we are reasonably confident that our simulations conserve magnetic helicity.

Moving to the case with VF boundary conditions, the lower panel of Figure 7 again shows the current helicity and the now-gauge dependent quantity  $H = \langle \mathbf{A} \cdot \mathbf{B} \rangle$ . Because of the presence of the gauge in the second term of Equation 6, this quantity is not physically meaningful itself. However, it is well-defined and provides a clue as to the behavior of the system. Once again, we inte-

grate  $-2\eta C$  using  $C$  in the same way. We expect that if magnetic helicity ejection occurs as a result of MRI turbulence, the value of  $H$  resulting from this integration will *not* track the actual value from the simulation, and indeed the lower panel of Figure 7 shows that it does not. Because of our gauge choice, we know that the flux of magnetic helicity flux density must be vertical, and this figure confirms that significant amounts of helicity escape. Furthermore,  $H$  fluctuates considerably more frequently in the VF case than in the run with periodic boundary conditions, showing that the timescale for variation of the global magnetic helicity decreases considerably when a helicity flux is allowed.

How does transport work in the periodic case, when there is no boundary flux? The only way for *net* helicity to change in this case is via resistive effects. Typically, resistively limited systems lead to lead to catastrophic quenching of the dynamo effect, where saturated magnetic energy is  $\propto \text{Rm}^{-1/2}$  (Vainshtein & Cattaneo 1992). We have shown that this is not the case in the stratified MRI: boundary conditions do not make significant differences in the transport (Figure 8) and for vertical field boundaries, Figure 4 demonstrates that the saturated large-scale field strength does not decline strongly with increasing  $\text{Rm}$ . Vishniac & Cho (2001) suggested a potential solution to this situation: helicity may not need to be ejected *entirely* across a system boundary (as in, say, a coronal mass ejection in the Solar dynamo). It could instead be transported *spectrally*, transferring from small scales to large. Indeed, Brandenburg (2001) show that exactly this occurs for a helically driven turbulence simulation with no shear or rotation—the prototypical  $\alpha^2$  dynamo. However, in this case, while equipartition fields are built even for periodic boundary conditions, the timescale required to do so is  $t_{\text{sat}} \propto \eta^{-1}$ : instead of catastrophic quenching for low resistivity, there is a catastrophic timescale problem instead (Brandenburg 2001). While our data is not conclusive on the relationship between  $\text{Rm}$ ,  $\text{Pm}$ , and the cycle period of large scale magnetic fields, it certainly does not suggest an inverse relationship between  $t_{\text{sat}}$  and  $\text{Rm}$ .

The two runs in Figure 7, taken together, tell an intriguing tale: the stratified MRI can transport angular momentum and build large scale magnetic energy even without a global helicity flux, but if one is allowed, the system does transport helicity across the domain boundary. Ultimately, as any boundary condition is in some sense an approximation of reality, we must understand the details of the accretion disk dynamo independent of the choice of such conditions.

The MRI is difficult to analyze in terms of a simple, two-scale mean field model: it does not present a single energy injection scale at which we could expect small scale helicity to be generated (Davis et al. 2010). This lack of clear scale separation makes the dynamo coefficients extracted via the test field method rather noisy and somewhat difficult to interpret, though it has been performed for the MRI by several groups (Brandenburg 2005; Gressel 2010). We forgo the test field method here, since we should be able to see a crude difference between the current helicity at the small and large scales if this is in fact what allows dynamo action, and thus transport. It is also clear comparing Figures 7 and 4 that the

timescale for building mean fields for the vertical field boundary condition case is not related in any obvious way to the flux of magnetic helicity. Furthermore, we do not see any significant changes in angular momentum transport with differing boundary conditions that do and do not allow a helicity flux. We address this in the next section.

#### 4.3. Helicity power spectra

We want to understand if scale transfer is responsible for dynamo action in closed, stratified MRI simulations. To do so, we compute the spectrum for the relative helicity,  $\mathcal{H}$ , on spherical shells in  $k$ -space, in runs with closed boundary conditions (i.e., periodic or perfect conductor). This ensures magnetic helicity is gauge independent by removing the surface terms in Equation 6.

In Figure 9, we show the helicity spectrum for  $\text{Re} = 3200$ ,  $\text{Pm} = 2$ . The field is not strongly helical at any scale, reaching only  $\mathcal{H} \sim 0.2$  at the smallest scales and remaining much lower on the largest scales. By contrast, the  $\alpha^2$  dynamo driven by a fully helical fluid forcing function has  $\mathcal{H} \sim 1$  at all scales (Brandenburg 2001). In that case, something similar to the classic inverse cascade of magnetic energy due to helicity conservation (Frisch et al. 1975; Pouquet et al. 1976, and § 4.2) occurs. In our case, however, we do not see a significant difference in the properties of the dynamo when the boundary conditions change between those that do not allow a flux of magnetic helicity and those that do, despite the fact that our results suggest that the MRI does in fact eject helicity when given the chance (see Figure 7). This resolves that observation: the MRI generated field is not strongly helical, even when both kinetic and magnetic  $\alpha$  effects occur. Thus, the constraints placed on the field by the conservation of magnetic helicity do not dominate its formation.

Finally, the fact that the relative helicity is peaked at small scales suggests that helicity constraints might be more important in the unstratified MRI, as suggested by Käpylä & Korpi (2010). In that case, the generation of field is not affected by dynamo waves and magnetic buoyancy, and so the helical turbulence might wind the field into a helically limited state. The small scales in Figure 9, where coherent dynamo waves and magnetic buoyancy effects are less important, show a rise in helicity, consistent with this idea.

#### 4.4. Future Work and Some Speculations

There remains some incongruity between our results for  $\alpha_M(z)$  and those of Gressel (2010). The only significant difference between his simulations and ours is his use of outflow boundary conditions on the fluid (the magnetic field boundary conditions are identical) and his use of a slightly larger domain. By running a model with a larger vertical extent, we have shown that above the  $\pm 2H$   $z$ -boundaries of our standard domain, the profile reverses and matches Gressel's. However, our results show a much stronger reversal near the midplane than his does, and this appears to be robust regardless of the domain size. The MRI coupled with outflow fluid boundary conditions leads to a magnetized wind, and this, together with the increased efficiency of the Parker instability with height, likely explains the remaining discrepancy. The role of *fluid* boundary conditions (which

we have not varied here) should also be examined, in order to better understand how winds interact with a disk dynamo (e. g. Vishniac & Cho 2001).

The overall pattern of dynamo generated fields in the  $z - t$  plane for the MRI is determined by the boundary conditions (Brandenburg et al. 1995), and this can be understood in terms of analytic mean-field theory (Brandenburg & Subramanian 2005). Nonetheless, Figure 8 shows that these various dynamo modes do not affect angular momentum transport. Thus, all of the boundary conditions studied here for the stratified MRI do an equally good job of generating magnetic flux for the MRI to continue to grow on. We have identified the fact that the magnetic fields are not fully helical at any scale as a likely reason why the boundary conditions do not determine the total angular momentum transport. Future work should address this point in more detail. Measuring the helicity spectrum in unstratified shearing boxes would determine if the dramatic differences between periodic and VF boundary conditions in that case are indeed related to a helicity-limited field evolution. Further theoretical work on the relationship between the mean field dynamo mechanisms in unstratified domains (likely the incoherent- $\alpha$  effect of Vishniac & Brandenburg 1997) and stratified domains (the  $\alpha - \Omega$  effect) and their expected degree of helicity is also required in order to complete this picture of dynamo-mediated MRI transport in the zero-net flux case.

We have demonstrated that understanding angular momentum transport via the MRI is strongly tied to the details of MHD turbulence and dynamo action. It is worth commenting briefly on some results for isotropic (non-shearing), non-rotating MHD turbulence, a much better studied system. Recently, several groups have demonstrated that non-local interactions in  $k$ -space are important at large scales in MHD turbulence, cross-coupling large scale velocity fields with small scale magnetic fields (e.g. Mininni et al. 2005; Lessinnes et al. 2009; Cho 2010). If such an analysis holds for the stratified MRI dynamo, it could explain the existence of a critical magnetic Reynolds number  $\text{Rm}_{\text{crit}}$ : if the large scale velocity fields are coupled to the magnetic dissipation range, a much more efficient energy sink appears than if they are coupled to a magnetic inertial range. In the latter case, the large scale velocities could contribute to small scale helicity production, continuing to drive large scale dynamo action. This could be verified by a shell-transfer analysis of the stratified MRI, which we will pursue in a future publication.

It is possible that our hyperdiffusive terms might be biasing our results. However, in a series of small-scale dynamo simulations, groups led by Schekochihin and Brandenburg have found no evidence that hyperviscosity plays a significant role (Schekochihin et al. 2005, 2007). There is a well-understood mechanism by which hyper-resistivity causes an increase in saturated mean field strengths for  $\alpha^2$  dynamos (Brandenburg & Sarson 2002). The main effect of hyperresistivity is simply to modify the timescales and length scales over which resistive effects occur, leading to differences from the magnetic helicity theory developed for Laplacian resistivity (Brandenburg 2001). Once the hyperresistive scales are properly accounted for, magnetic helicity conservation indeed correctly predicts the saturation field strength.

Thus, our results are unlikely to be significantly affected by this mechanism, as we do not offer a detailed theory for the saturation amplitude for the MRI. Furthermore, the Brandenburg & Sarson (2002) simulations did not include a standard, Laplacian resistivity term, while ours do.

More directly, our resolution study also suggests that hyperdiffusion is not a dominant effect: for integrated quantities, our 64 zones/ $H$  and 128 zones/ $H$  runs are converged (see Figure 2 and Table 1). Because the hyperdiffusive coefficients are smaller for larger resolution, this suggests that the values of the coefficients make little difference. Nevertheless, we cannot rule out any *systematic* hyperdiffusive influence on the dynamo behavior without running simulations at high enough resolution to simultaneously resolve the MRI and dissipate energy fast enough at the grid scale using only regular Laplacian diffusion operators at all times and through the entire spatial domain. While this is practical for unstratified MRI simulations, because the distribution of  $u$  is relatively narrow, in stratified simulations where the distribution of velocities is much broader, it is hard to maintain stability without hyperdiffusivity. Direct numerical simulations of stratified MRI thus require very high resolution and are therefore extremely expensive and are not practical for the parameter study considered here.

## 5. CONCLUSIONS

We have demonstrated that the strength of the angular momentum transport parameter  $\alpha_{SS}$  in stratified, MRI driven turbulence appears independent of the magnetic Prandtl number  $Pm$  above some critical magnetic Reynolds number  $Rm_{crit} \sim 3000$ . Our models suggest that the value of  $Pm$  at which the flattening of the  $Pm - \alpha_{SS}$  relation occurs is a function of  $Rm$ . Käpylä & Korpi (2010) demonstrated that in the unstratified case,  $\alpha_{SS}$  is independent of  $Pm$  if boundary conditions allow for the ejection of magnetic helicity. If these conclusions are indicative of the asymptotic state in real disks, then concerns about  $Pm$ -dependent scaling of the MRI in real disks, which are certainly stratified, would be entirely alleviated if the  $Rm$  is sufficiently high.

We find that although the stratified MRI does eject magnetic helicity when boundary conditions allow, the fields it generates are not significantly helical at any scale or at any time during their evolution. We suggest that this points toward a theoretical explanation for why previous stratified MRI simulations with periodic boundary conditions in the vertical direction sustained dynamo action while unstratified simulations at similar  $Pm$  did not: while the *unstratified* MRI dynamo can only grow by ejecting helicity, the *stratified* MRI dynamo does not wind itself into a fully helical state and is thus not limited by the transport of magnetic helicity across boundaries. This is only a preliminary result; future work should follow in more detail evolution of magnetic helicity in both

unstratified and stratified MRI simulations. The scaling results presented in this paper appear to be the result of mean field dynamo action: when  $Rm > Rm_{crit}$ , the dynamo is ordered and angular momentum transport does not depend on  $Pm$ ; when  $Rm < Rm_{crit}$ , the dynamo is disordered and angular momentum transport is strongly dependent on  $Pm$ . This underlines the importance of understanding the MRI dynamo system in order to understand the details of angular momentum transport in accretion disks, as pointed out by Blackman (2010) and Gressel (2010). In order to make more concrete predictions about the structure and evolution of disks, we will need to turn to global simulations. Given that global simulations inevitably involve much lower resolution than local simulations, they should involve some kind of sub-grid model in order to properly capture small-scale dynamics. This work here provides a preliminary step toward such a model.

Astrophysical disks have tremendous values of  $Re$  and  $Rm$ , so even with a  $Pm \sim 10^{-8}$ ,  $Rm \gg Rm_{crit}$  if  $Re \sim 10^{12}$ . In order to make this concrete, we use the Balbus & Henri (2008) estimates for the viscosity and resistivity to estimate the Reynolds number for a typical protoplanetary disk active region. Using a fiducial midplane density  $\rho \simeq 10^{-10} \text{ g cm}^{-3}$ , scale height  $H \simeq 0.05 \text{ AU}$ , and temperature  $T \sim 500 \text{ K}$  (Ilgner & Nelson 2006), we arrive at  $Re \simeq 4 \times 10^{16}$ . The same estimates give  $Pm \simeq 1 \times 10^{-8}$ , easily fulfilling  $Rm \gg Rm_{crit}$  and so we expect the transport to be independent of  $Pm$  in the *active* regions of such disks.

A major caveat to this, of course, is that the ionization state of such disks is not well established. Our results do not bear on the question of dead zones, which are regions of high resistivity resulting from poor ionization rather than low temperatures. Nevertheless, our point here is to establish the dynamo state of the active region, *not* to compute detailed predictions for  $\alpha_{SS}$  in such systems. Finally, black hole accretion systems should also have  $Rm \gg Rm_{crit}$ : the same computation yields  $Re \simeq 6 \times 10^{12}$  and  $Pm \simeq 0.1$  at roughly 100 Schwarzschild radii from a  $10 M_{\odot}$  black hole (Balbus & Henri 2008).

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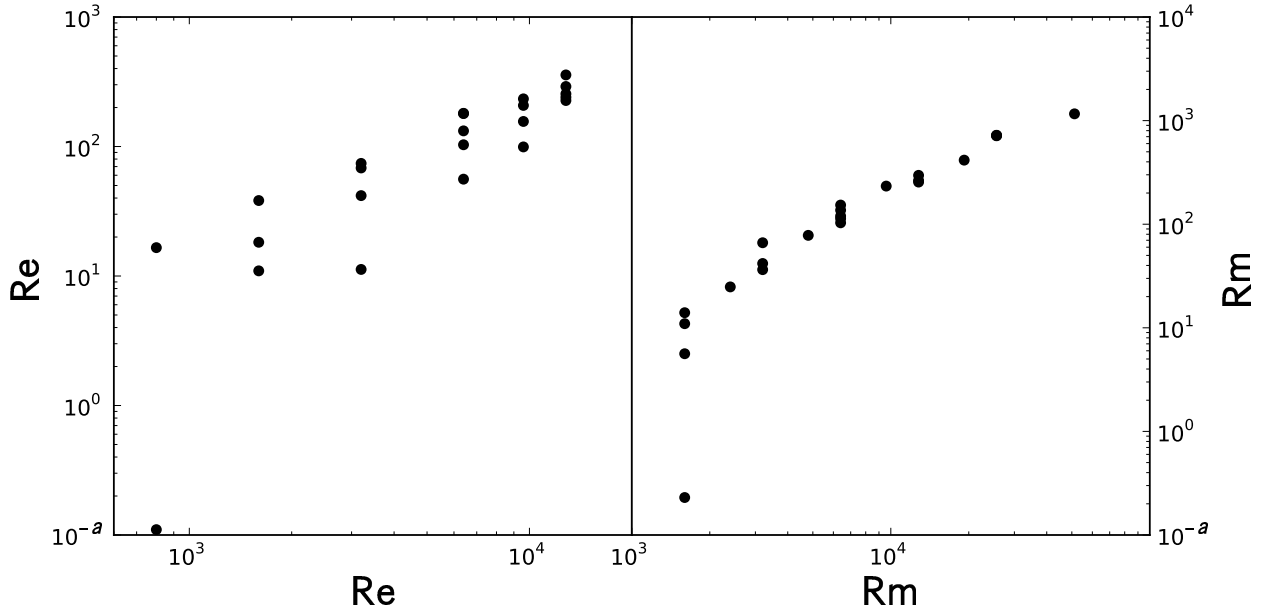
## APPENDIX

### THE KEPLER GAUGE

We here derive the Kepler gauge used in our definition of the magnetic vector potential. Beginning from the induction equation for the vector potential,

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \eta \mu_0 \mathbf{J} \quad (\text{A1})$$





**Figure 1.** (left) Reynolds number parameter  $Re$  versus the ratio of advection to viscous dissipation in the simulations. (right) Magnetic Reynolds number parameter  $Rm$  versus the ratio of advection to resistive dissipation in the simulations.

where  $\mathbf{J} = \nabla \times \mathbf{B}$ , we expand the velocity  $\mathbf{v} = \mathbf{u} + \mathbf{U}^{\text{kep}}$ , where  $\mathbf{U}^{\text{kep}} = q\Omega_0 x \hat{\mathbf{y}}$  is the linearized Keplerian shear velocity. Making this substitution, we have

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} + [\mathbf{U}^{\text{kep}} \times \mathbf{B}] - \eta \mu_0 \mathbf{J} \quad (\text{A2})$$

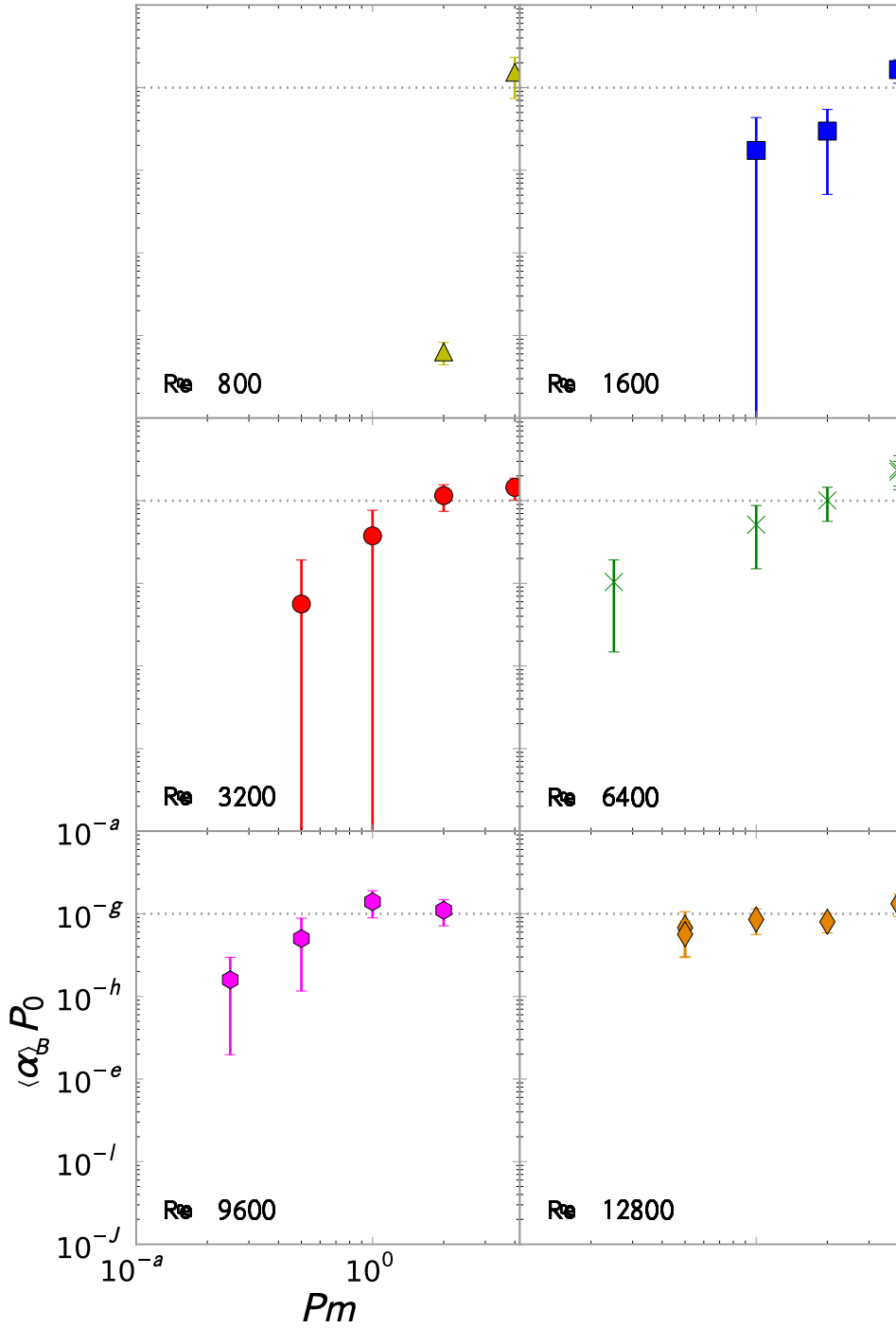
Expanding the second term on the right hand side,

$$\mathbf{U}^{\text{kep}} \times \mathbf{B} = \mathbf{U}^{\text{kep}} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{U}^{\text{kep}} \cdot \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{U}^{\text{kep}}) - \mathbf{U}^{\text{kep}} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{U}^{\text{kep}}. \quad (\text{A3})$$

The first term on the right hand side is the gradient of a scalar function  $\phi_{\text{kep}} = \mathbf{U}^{\text{kep}} \cdot \mathbf{A}$ , which we term the Kepler gauge. The remainder of the terms represent shear and advection, and correspond to the second terms on either side of Equation 1. We refer the reader to Hubbard & Brandenburg (2011) for a generalization of this gauge to any advective velocity.

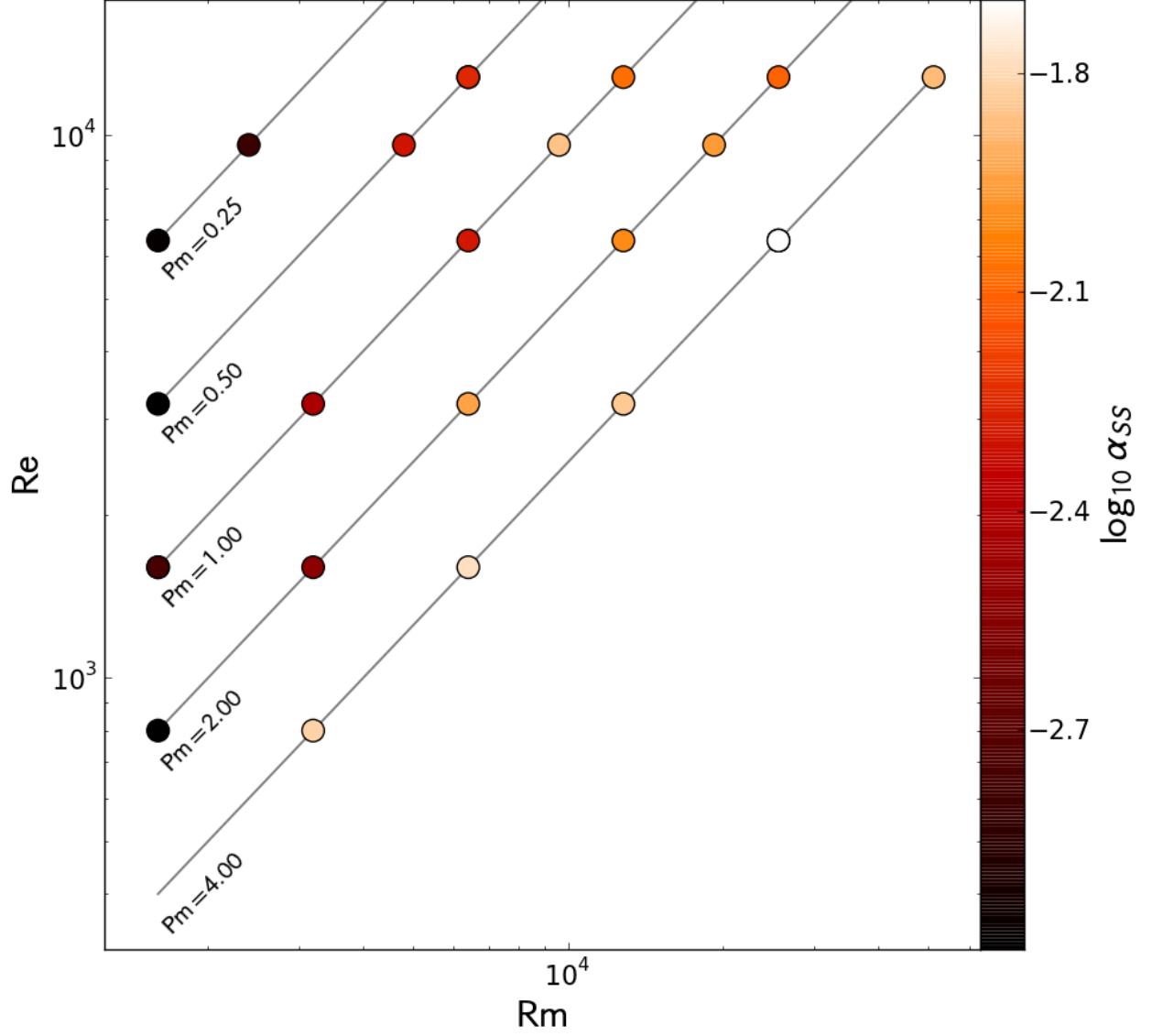
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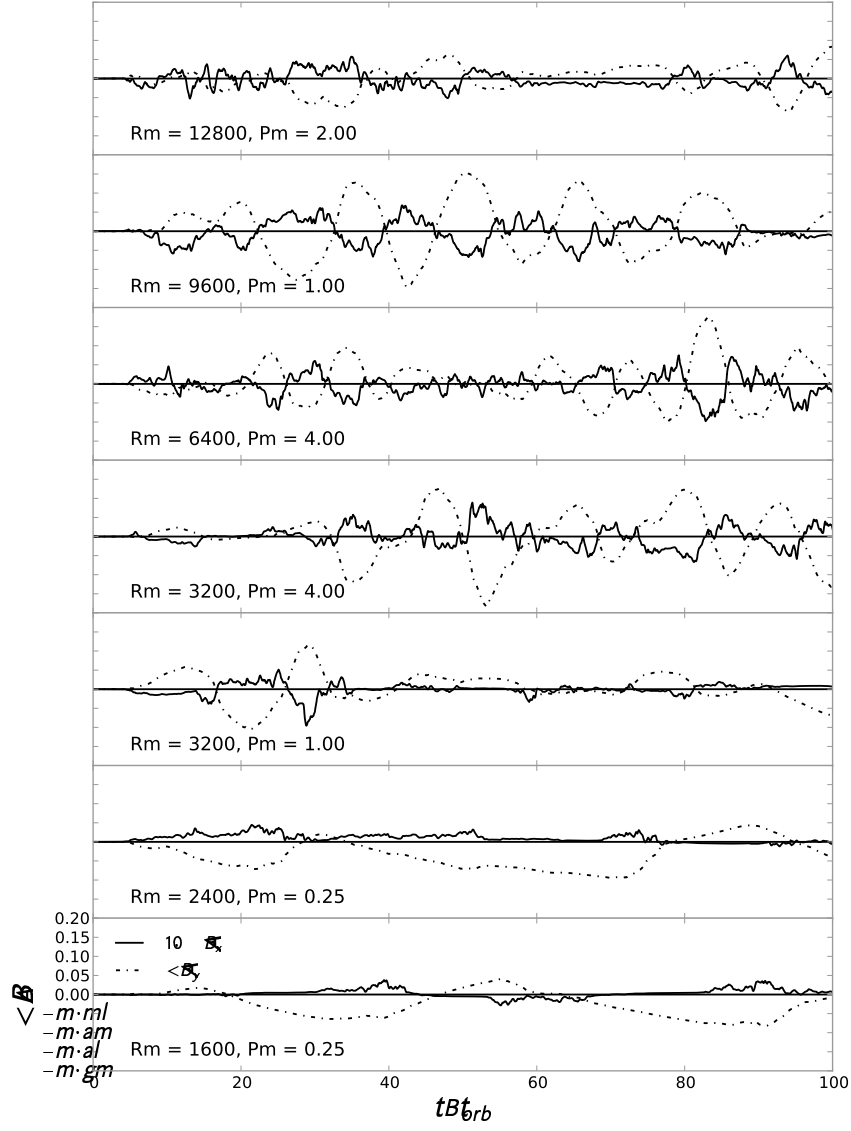
**Figure 2.** Volume and time averaged value of the angular momentum transport parameter  $\alpha_{SS}$  as a function of magnetic Prandtl number  $Pm$ . Each subfigure is plotted on the same axes, but shows models with different Reynolds number  $Re$ . Angular momentum transport increases with  $Pm$  up to a threshold value of  $Pm$  beyond which it remains constant. This threshold occurs at lower  $Pm$  for increasing  $Re$ , suggesting a constant threshold at some critical magnetic Reynolds number  $Rm_{crit}$ . The superposed points in the  $Re = 6400$ ,  $Re = 9600$  (not visible), and  $Re = 12800$  series represent resolution studies with one point representing a model with 128 zones/ $H$ , double the standard resolution.

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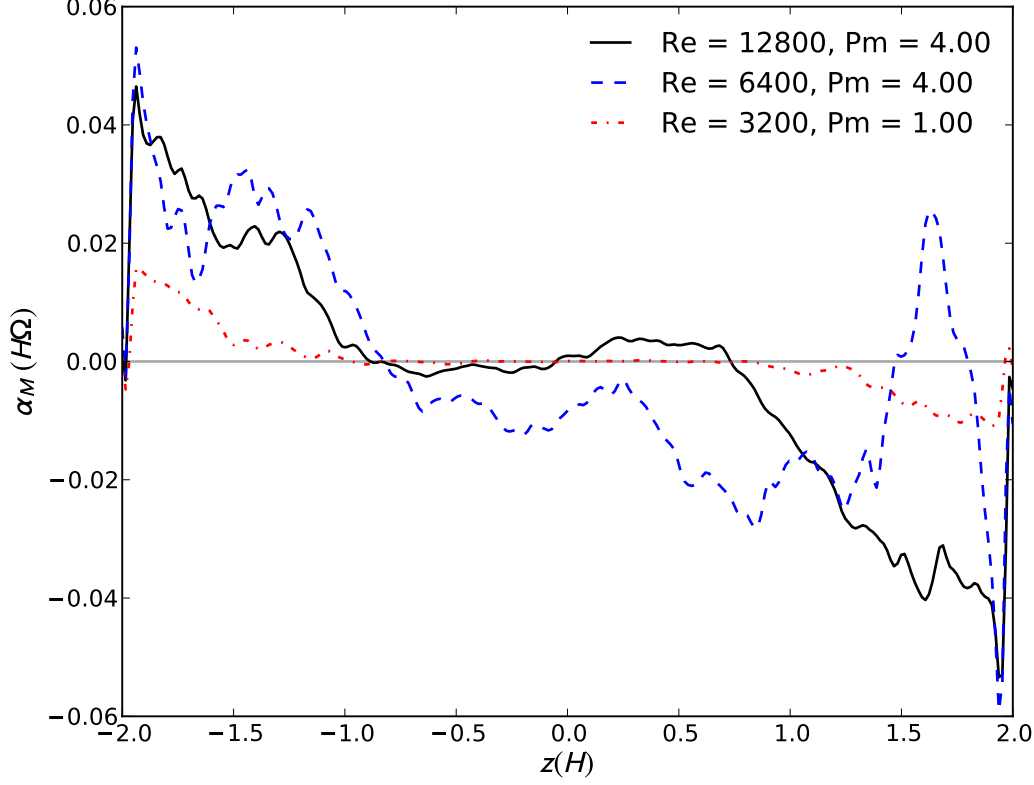


**Figure 3.** The log of  $\alpha_{SS}$  as a function of both viscous and magnetic Reynolds numbers  $Rm$  and  $Re$ . Darker shades represent weaker angular momentum transport. Below a critical magnetic Reynolds number  $Rm_{crit}$  slightly above  $10^3$ , transport falls off significantly.

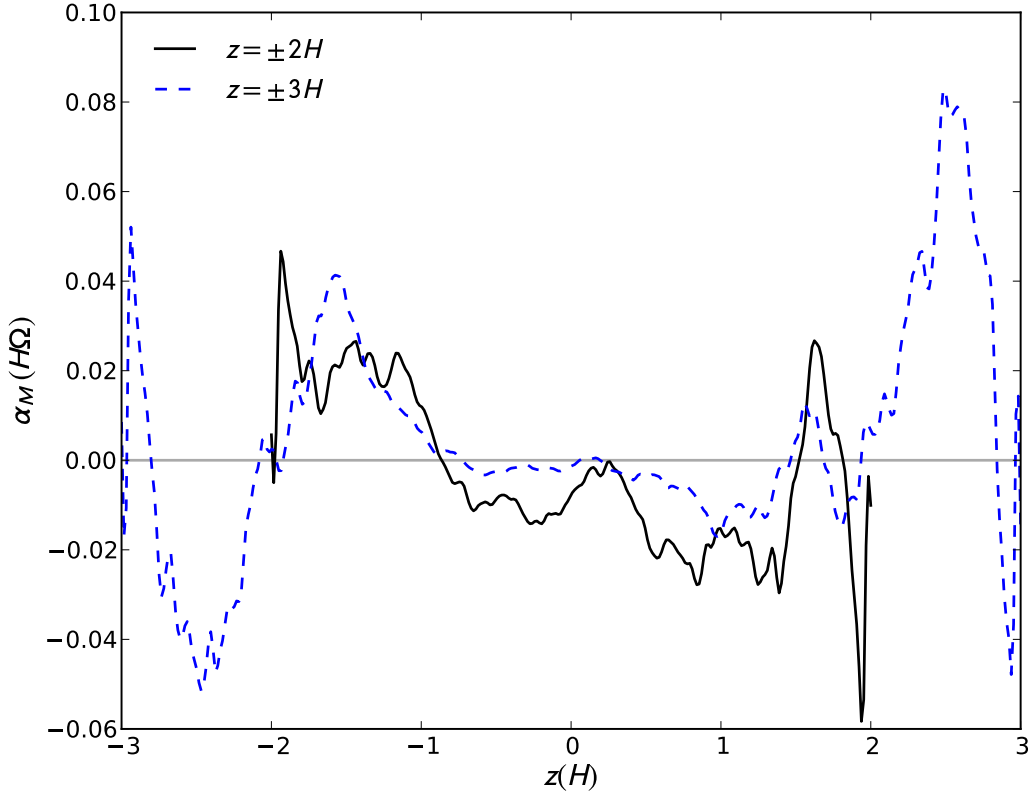
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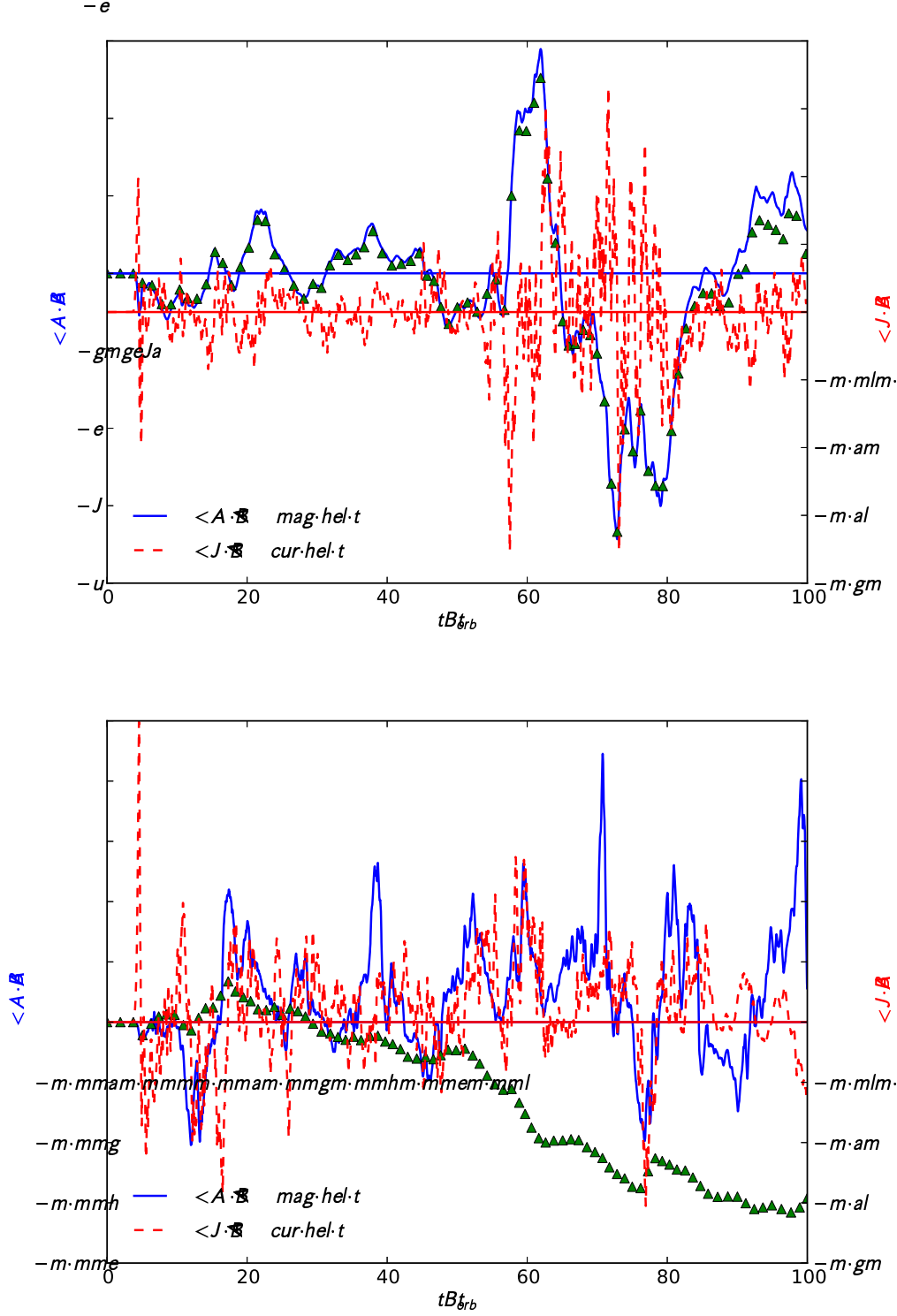
**Figure 4.** Mean fields,  $\langle B_x \rangle$  and  $\langle B_y \rangle$ , as a function of time for seven simulations with  $\langle B_x \rangle$  multiplied by a factor of 10 for clarity. In all cases, the  $x$  and  $y$  components are out of phase, but there are two distinct dynamo modes, exemplified by the  $Rm = 2400, Pm = 0.25$  and  $Rm = 3200, Pm = 4$  cases. All vertical axes for  $\langle B \rangle$  are on the same scale given in the bottom panel.



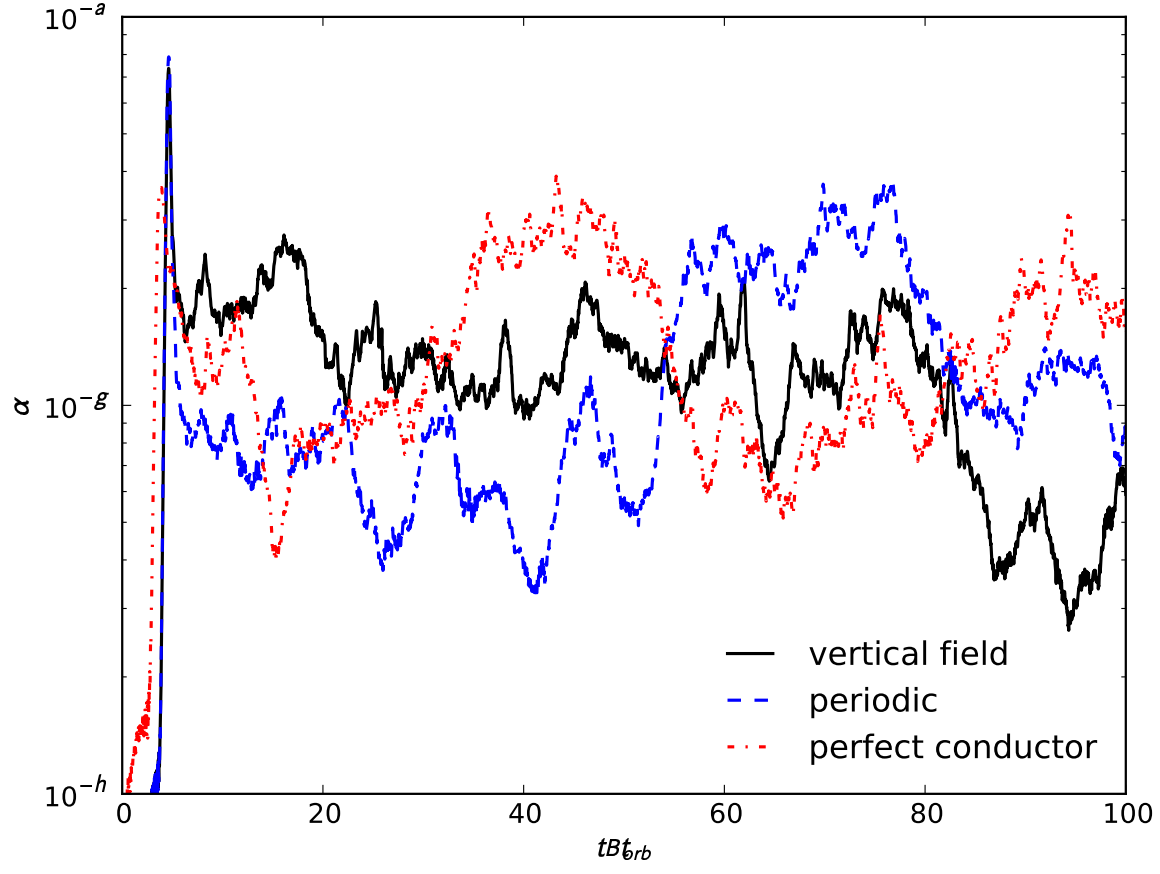
**Figure 5.** Radial and azimuthally averaged  $\langle \alpha_M \rangle_{xy}(z)$ , for several runs averaged over  $t > 20t_{orb}$ . Note the profile is opposite in sign to that presented in Gressel (2010).



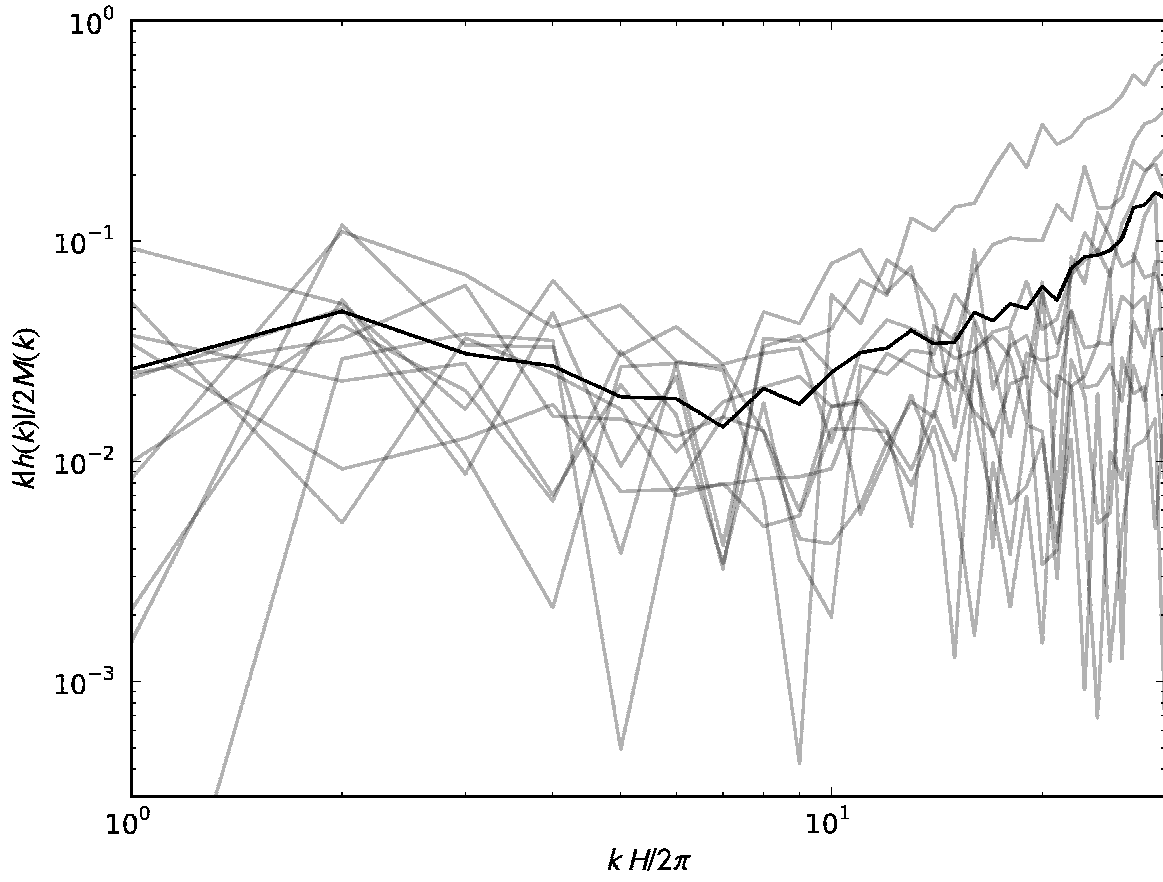
**Figure 6.** Radial and azimuthally averaged  $\langle \alpha_M \rangle_{xy}(z)$ , for two runs with  $Re = 6400$  and  $Pm = 4$  averaged over  $t > 20t_{orb}$ . One model has the standard vertical extent,  $|z| < 2H$  (solid line), while the other has an extended vertical extent,  $|z| < 3H$  (dashed line). The sign reversal seen above  $|z| \sim 2$  may explain the apparent discrepancy between our results and those of Gressel (2010).



**Figure 7.** Magnetic helicity  $H = \langle \mathbf{A} \cdot \mathbf{B} \rangle$  (blue solid line) and current helicity  $C = \langle \mathbf{J} \cdot \mathbf{B} \rangle$  (red dashed line), as a function of time for two runs with  $\text{Re} = 3200$  and  $\text{Pm} = 2$ , one with periodic boundary conditions (upper panel; ensuring that  $H$  is gauge independent and thus physically meaningful) and one with VF boundary conditions. The green triangles are a simple time integration of  $-2\eta C$ , plotted every 100 timesteps. Horizontal lines mark the zero points for each axis. For periodic boundary conditions, the integration nearly overlies  $H$ , demonstrating that magnetic helicity is indeed constrained by resistive action on current helicity. For VF boundary conditions, the integration (green triangles) does not track the helicity at all, because the flux terms in Equation 6 are not zero in this case.



**Figure 8.** Volume averaged  $\langle \alpha \rangle$  for a  $\text{Pm} = 2$ ,  $\text{Re} = 3200$  model with vertical field, periodic, and perfect conductor boundary conditions. The transport is comparable regardless of boundary conditions.



**Figure 9.** Relative magnetic helicity  $\mathcal{H} = kH/2M$  as a function of scale for a run with  $\text{Re} = 3200$ ,  $\text{Pm} = 2$  and periodic boundary conditions, averaged over a  $10t_{orb}$  period in saturation. Light lines show individual timesteps, the dark line is the average. The value remains well below unity at all scales, showing that the field is not significantly helical at any scale.



Table 1

Run	Re	Rm	Pm	$L_z$ (H)	$\nu_3, \eta_3, D_3$	zones/H	Re'	Rm'	$u_{rms}$	$B^2/2$
A	12800	6400	0.5	4	3.0(−13)	128	226.58	113.29	0.17	0.03
B	12800	6400	0.5	4	9.6(−12)	64	239.96	119.98	0.18	0.03
C	12800	12800	1	4	9.6(−12)	64	255.15	255.15	0.19	0.04
D	12800	25600	2	4	9.6(−12)	64	357.16	714.32	0.26	0.05
E	12800	51200	4	4	9.6(−12)	64	290.71	1162.86	0.21	0.06
F	9600	19200	2	4	3.0(−13)	128	207.70	415.40	0.20	0.05
G	9600	9600	1	4	9.6(−12)	64	233.36	233.36	0.23	0.07
H	9600	4800	0.5	4	9.6(−12)	64	156.13	78.06	0.15	0.02
I	9600	2400	0.25	4	9.6(−12)	64	99.29	24.82	0.10	0.01
J	6400	12800	2	4	9.6(−12)	64	132.10	264.18	0.19	0.05
K	6400	6400	1	4	9.6(−12)	64	103.18	103.18	0.15	0.02
L	6400	25600	4	4	9.6(−12)	64	180.42	721.68	0.27	0.13
Ll	6400	25600	4	6	9.6(−12)	64	193.20	772.81	0.28	0.06
M	6400	25600	4	4	3.0(−13)	128	179.70	718.79	0.26	0.10
N	6400	1600	0.25	4	9.6(−12)	64	55.96	13.99	0.08	0.01
O	3200	12800	4	4	9.6(−12)	64	74.08	296.31	0.22	0.06
P	3200	6400	2	4	9.6(−12)	64	68.31	136.62	0.20	0.05
Q	3200	3200	1	4	9.6(−12)	64	41.77	41.77	0.12	0.02
R	3200	1600	0.5	4	9.6(−12)	64	11.24	5.62	0.03	0.01
S	1600	6400	4	4	9.6(−12)	64	38.27	153.07	0.23	0.08
T	1600	3200	2	4	9.6(−12)	64	18.22	36.43	0.11	0.02
U	1600	1600	1	4	9.6(−12)	64	10.96	10.96	0.06	0.02
V	800	3200	4	4	9.6(−12)	64	16.56	66.24	0.20	0.09
W	800	1600	2	4	9.6(−12)	64	0.11	0.23	0.00	0.00

**Note.** — Basic run parameters.  $\text{Re}' = u_{rms}/\nu k_1$  and  $\text{Rm}' = u_{rms}/\eta k_1$ , where  $k_1 = 2\pi/4H$  is the smallest integer wavenumber in the box, measure the ratio of turbulent advection to viscous and magnetic diffusion, respectively.  $\eta_3, \nu_3$ , and  $D_3$  are the hyperresistive, hyperviscous, and hyperdiffusive coefficients.